

# ADÈLE RESIDUE SYMBOL AND TATE'S CENTRAL EXTENSION FOR MULTILOOP LIE ALGEBRAS

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**ABSTRACT.** We generalize the linear algebra setting of Tate's central extension to arbitrary dimension. In general, one obtains a Lie  $(n+1)$ -cocycle. We compute it explicitly. The construction is based on a Lie algebra variant of Beilinson's adelic multidimensional residue symbol, generalizing Tate's approach to the local residue symbol for 1-forms on curves.

Firstly, recall that to every Lie algebra  $\mathfrak{g}$  one can associate its loop Lie algebra  $\mathfrak{g}[t^{\pm}]$ . Iterating this construction, we obtain so-called *multiloop Lie algebras*,  $\mathfrak{g}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ .

To begin with, we show that various classes of interesting multiloop Lie algebras can all be embedded into a large (infinite-dimensional) Lie algebra:

**Theorem 1.** *For every field  $k$  and integer  $n \geq 1$  there is a universal Lie algebra  $\mathfrak{G}$  naturally containing the following:*

- (1) *the abelian Lie algebra  $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ ,*
- (2) *Lie algebras of derivations, i.e. spanned by*

$$t_1^{s_1} \cdots t_n^{s_n} \partial_{t_i}, \quad (\text{acting on } k[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$$

- (3) *for any finite-dimensional simple Lie algebra  $\mathfrak{g}$  the multiloop algebra*

$$\mathfrak{g}[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

*The universal Lie algebra  $\mathfrak{G}$  comes with a canonical Lie  $(n+1)$ -cocycle  $\phi \in H^{n+1}(\mathfrak{G}, k)$ . For  $n = 1$  this cocycle determines a central extension (known as Tate's central extension)*

$$0 \longrightarrow k \longrightarrow \widehat{\mathfrak{G}} \longrightarrow \mathfrak{G} \longrightarrow 0$$

*and the pullback of it to one of the above types of subalgebras yields (respectively)*

- (1) *the Heisenberg algebra,*
- (2) *the Virasoro algebra,*
- (3) *the affine Lie algebra  $\widehat{\mathfrak{g}}$  associated to  $\mathfrak{g}$ .*

This will be stated in more detail and proven in §4. From a classical point of view it is surprising that all these popular examples of central extensions are all induced by one universal cocycle. For  $n = 1$  the above is well-known, see for example [3, §2.1]. For  $n = 1, 2$  see [7]. In the language of the latter,  $\mathfrak{G}$  is an example of a “master Lie algebra”.

We are interested in the nature of  $\phi$  for  $n > 1$  – even if such cocycles cannot be

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interpreted as a central extension anymore (we get crossed modules, etc.). Indeed, they are meaningful, as we shall see.

A key point of this text is the actual computation of  $\phi$ :

**Theorem 2.** *The cocycle  $\phi \in H^{n+1}(\mathfrak{G}, k)$  is given explicitly by*

$$\begin{aligned} & \phi(f_0 \wedge f_1 \wedge \dots \wedge f_n) \\ &= \text{tr} \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_{\pi(1)}) P_1^{\gamma_1}) \\ & \quad \dots (P_n^{-\gamma_n} \text{ad}(f_{\pi(n)}) P_n^{\gamma_n}) f_0, \end{aligned}$$

where  $P_1^+, \dots, P_n^+$  are certain commuting idempotents (see §3-4 for details).

The proof and details regarding the  $P_i^\pm$  can be found in §3-4. The formula is rather complicated. However, the pullback to particular subalgebras of  $\mathfrak{G}$  can be much nicer, for example for multiloop Lie algebras of simple Lie algebras, we get the following:

**Theorem 3.** *Suppose  $\mathfrak{g}/k$  is a finite-dimensional centreless Lie algebra (e.g. simple). For  $Y_0, \dots, Y_n \in \mathfrak{g}$  we call*

$$B(Y_0, \dots, Y_n) := \text{tr}_{\text{End}_k(\mathfrak{g})}(\text{ad}(Y_0) \text{ad}(Y_1) \dots \text{ad}(Y_n))$$

the ‘generalized Killing form’. Then the pullback of  $\phi$  to  $\mathfrak{g}[t_1^\pm, \dots, t_n^\pm]$  is explicitly given by

$$\begin{aligned} & \phi(Y_0 t_1^{c_{0,1}} \dots t_n^{c_{0,n}} \wedge \dots \wedge Y_n t_1^{c_{n,1}} \dots t_n^{c_{n,n}}) = \\ & (-1)^n \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) B(Y_{\pi(1)}, \dots, Y_{\pi(n)}, Y_0) \prod_{i=1}^n c_{\pi(i), i} \end{aligned}$$

whenever  $\forall i \in \{1, \dots, n\} : \sum_{p=0}^n c_{p,i} = 0$  and zero otherwise. Here  $c_{i,p} \in \mathbf{Z}$  for all  $i = 0, \dots, n$  and  $p = 1, \dots, n$ .

If  $\mathfrak{g}$  is finite-dimensional simple and  $n = 1$ , then the class  $\phi$  yields the universal central extension of the loop Lie algebra  $\mathfrak{g}[t_1, t_1^{-1}]$ , the associated affine Lie algebra  $\widehat{\mathfrak{g}}$  (without extending by a derivation),

$$0 \longrightarrow k \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g}[t_1, t_1^{-1}] \longrightarrow 0.$$

In this case  $B$  is obviously just the ordinary Killing form of  $\mathfrak{g}$ . The above theorem will be proven in §6.

Additionally, we should say that these computations have an application outside the theory of Lie algebras.

For this we need to return to the roots of the subject. In 1967 J. Tate [14] showed that the residue of a rational 1-form  $f dg$  at a closed point  $x$  on an algebraic curve  $X/k$  can be expressed as a certain operator-theoretic trace on an infinite-dimensional space. Arbarello, de Concini and Kac [1, eq. (2.7)] reformulated this as

$$(0.1) \quad \text{res}_x f dg = \text{tr}([\pi, g]f).$$

On the right-hand side the functions  $f, g$  are to be read as multiplication operators acting on the local field  $\text{Frac } \widehat{\mathcal{O}}_{X,x} \simeq \kappa(x)((t_1))$ , seen as a  $\kappa(x)$ -vector space, and  $\pi$  denotes some projector on the non-principal part, i.e. “we cut off the principal part of the Laurent series”. It is natural to ask whether there exists a generalization of this formula to higher residues. We can give such a formula; it will be proven in §5:

**Theorem 4.** *For a multiple Laurent polynomial ring with residue field  $k$ , say*

$$R := k[t_1^\pm, \dots, t_n^\pm],$$

*and  $f_0, \dots, f_n \in R$  we have*

$$\begin{aligned} & \text{res}_{t_1} \cdots \text{res}_{t_n} f_0 df_1 \cdots df_n \\ &= (-1)^n \text{tr} \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_{\pi(1)}) P_1^{\gamma_1}) \\ & \quad \cdots (P_n^{-\gamma_n} \text{ad}(f_{\pi(n)}) P_n^{\gamma_n}) f_0, \end{aligned}$$

*where  $P_1^\pm, \dots, P_n^\pm$  are suitable projectors (explained in §5, eq. 5.3).*

- (1) *For  $n = 1$  and  $\pi := P_1^+$  the formula reduces to the familiar eq. 0.1, as in [1].*
- (2) *If we have  $f_i = t_1^{c_{i,1}} \cdots t_n^{c_{i,n}}$  for  $i = 0, \dots, n$ , the formula reduces to*

$$\text{res } f_0 df_1 \cdots df_n = \det \begin{pmatrix} c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \cdots & c_{n,n} \end{pmatrix} \quad \text{if } \forall i : \sum_{p=0}^n c_{p,i} = 0$$

*and the residue is zero if the condition on the right-hand side is not satisfied.*

- (3) *For  $n = 1$  and  $f_1 = t_1$  this reduces by linearity to the classical definition*

$$\text{res } \alpha t_1^{c_1} dt_1 = \begin{cases} \alpha & \text{if } c_1 = -1 \\ 0 & \text{if } c_1 \neq -1. \end{cases}$$

How to construct the cocycle  $\phi$ ?

There are various ways to approach this construction. Frenkel and Zhu [7] use distinguished generators of the cohomology ring of infinite matrix algebras, based on computations of Feigin and Tsygan [4]. This is a very natural approach. However, in this text we use a different approach based on Beilinson's multidimensional adelic residue [2]. Originally, this approach was only used to generalize Tate's approach to the residue symbol to several variables, but it readily generalizes to the problem we are discussing here. This might be interesting also since [2] does not give an explicit formula – and it is not totally trivial to extrapolate a formula from the definition:

**Theorem 5.** *The formula in Thm. 4 arises from the construction of Beilinson in the paper [2, Lemma 1], i.e. it is the composition*

$$(0.2) \quad \begin{aligned} \Omega_{R/k}^n & \xrightarrow{(-1)^n \varkappa} H_{n+1}^{\text{Lie}}(\mathfrak{G}, k) \xrightarrow{\rho_2} E_{0,n+1}^{n+1} \\ & \xrightarrow{(d_{n+1})^{-1}} E_{n+1,1}^{n+1} \xrightarrow{\rho_1} H_0^{\text{Lie}}(\mathfrak{G}, N_{n+1}) \xrightarrow{\text{tr}} k, \end{aligned}$$

*where*

- $\varkappa : f_0 df_1 \wedge \cdots \wedge df_n \mapsto f_0 \wedge \cdots \wedge f_n$ ,
- $N_{n+1}$  *is a certain  $\mathfrak{G}$ -module (see §3 for the definition, or  $T_{*N}$  in [2]), and*
- $\rho_1, \rho_2$  *are edge maps and  $d_{n+1}$  a differential on the  $(n+1)^{\text{th}}$  page of a certain spectral sequence  $E_{\bullet,\bullet}^\bullet$  (constructed in Lemma 2, or see [2, Lemma 1]).*

This result is only meaningful to readers familiar with the paper [2].

The above theorem actually lies at the heart of our approach. We formulate a contracting homotopy for (mild generalizations of) the relevant complexes in [2]

and then, in a slightly tedious computation, make the spectral sequence differential  $d_{n+1}$  explicit on the basis of this.

Finally, for applications in algebraic geometry, e.g. the interpretation as a local residue, it is unfortunate to interpret the word “loop Lie algebra” as  $\mathfrak{g}[t, t^{-1}]$ . It is better to work with Laurent series, i.e.  $\mathfrak{g}((t))$ , or even local components of adèles. Tate’s original work uses the language of adèles for example. For this reason, we shall axiomatize all these variations through the notion of a “cubically decomposed algebra” (essentially taken from [2], where it’s not given a name).

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**0.2. What is not here.** In the present text I only discuss the ‘linear algebra setting’ of Tate’s central extension ([3, §1] for the case  $n = 1$ ). There is also a ‘differential operator setting’ ([3, §2]), which I will treat in a future text. Roughly speaking,  $\mathfrak{G}$  will be replaced by much smaller algebras of differential operators on a vector bundle.

Moreover, I do not treat the true multiloop analogue of an affine Kac-Moody algebra in the present text. Already for  $n = 1$  I only consider the ‘plain’ affine Lie algebras without extending by a derivation. From the perspective of a triangular decomposition, this is a rather horrible omission: the root spaces are infinite-dimensional! However, as the reader can probably imagine from the computations in §5, §6 the calculation gets a lot more complicated in the presence of derivations. Thus, this aspect will also be deferred to a future text. The same applies to the analogue of the plain Virasoro algebra. There should also be a nonlinear analogue, distinguished cohomology classes for multiloop groups. The cases  $n = 1, 2$  (along with a higher representation theory in categories) are treated in detail by Frenkel and Zhu in [7].

One should also mention that there are completely orthogonal generalizations of Kac-Moody/Virasoro cocycles to multiloop Lie algebras, see for example [8, §9], [13].

## 1. BASIC FRAMEWORK

For an associative algebra  $A$  we shall write  $A_{Lie}$  to denote the associated Lie algebra.

**Definition 1** ([2]). *An  $(n$ -fold) cubically decomposed algebra (over a field  $k$ ) is the datum  $(A, (I_i^\pm), \tau)$ :*

- *an associative unital (not necessarily commutative)  $k$ -algebra  $A$ ;*
- *two-sided ideals  $I_i^+, I_i^-$  such that  $I_i^+ + I_i^- = A$  for  $i = 1, \dots, n$ ;*
- *writing  $I_i^0 := I_i^+ \cap I_i^-$  and  $I_{tr} := I_1^0 \cap \dots \cap I_n^0$ , a  $k$ -linear map*

$$\tau : I_{tr, Lie} / [I_{tr, Lie}, A_{Lie}] \rightarrow k.$$

For any finite-dimensional  $k$ -vector space  $V$  certain infinite matrix algebras act naturally on the  $k$ -vector space of multiple Laurent polynomials  $V[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ . This yields an example of this structure, see §1.1 below. There is also an analogue for  $V((t_1)) \cdots ((t_n))$ , which we leave to the reader to formulate (this links to higher local fields, see [6]). Local components of Parshin-Beilinson adèles of schemes yield another example, see [2, §1]. In *loc. cit.* the ideals  $I_i^+, I_i^-$  are called  $X^i, Y^i$ . The

latter gives the multidimensional generalization of the adèle formulation of Tate [14]. See [5], [9], [10], [12] for more background on higher-dimensional adèles and their uses.

**1.1. Infinite Matrix Algebras.** Fix a field  $k$ . Let  $R$  be an associative  $k$ -algebra, not necessarily unital or commutative. Define an algebra of infinite matrices

$$(1.1) \quad E(R) := \{\phi = (\phi_{ij})_{i,j \in \mathbf{Z}}, \phi_{ij} \in R \mid \exists K_\phi : |i - j| > K_\phi \Rightarrow \phi_{ij} = 0\}.$$

Define a product by  $(\phi \cdot \phi')_{ik} := \sum_{j \in \mathbf{Z}} \phi_{ij} \phi'_{jk}$ , the usual matrix multiplication formula; this sum only has finitely many non-zero terms and one can choose  $K_{\phi\phi'} := K_\phi + K_{\phi'}$ . Then  $E(R)$  becomes an associative  $k$ -algebra. If  $R$  is unital,  $E(R)$  is also unital.  $E$  is a functor from associative algebras to associative algebras; for a morphism  $\varphi : R \rightarrow S$  there is an induced morphism  $E(\varphi) : E(R) \rightarrow E(S)$  by using  $\varphi$  entry-by-entry, i.e.  $(E(\varphi)\phi)_{ij} := \varphi(\phi_{ij})$ . If  $I \subseteq R$  is an ideal (which is in particular a non-unital associative ring),  $E(I) \subseteq E(R)$  is an ideal. Moreover, for ideals  $I_1, I_2$  one has  $E(I_1 \cap I_2) = E(I_1) \cap E(I_2)$  and  $E(I_1 + I_2) = E(I_1) + E(I_2)$ , as a sum of ideals. Next, define

$$I^+(R) := \{\phi \in E(R) \mid \exists B_\phi : i < B_\phi \Rightarrow \phi_{ij} = 0\}$$

$$I^-(R) := \{\phi \in E(R) \mid \exists B_\phi : j > B_\phi \Rightarrow \phi_{ij} = 0\}$$

and one checks easily that  $I^+(R), I^-(R)$  are two-sided ideals in  $E(R)$ . The following figure attempts to visualize the shape of the matrices in  $E(R), I^+(R)$  and  $I^-(R)$  respectively:

$$\begin{bmatrix} * & * & & & \\ * & * & * & & \\ * & * & * & * & \\ & * & * & * & * \\ & & * & * & * & * \\ & & & * & * & * & * \\ & & & & * & * & * & * \\ & & & & & * & * & * & * \end{bmatrix} \quad \begin{bmatrix} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \quad \begin{bmatrix} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{bmatrix}$$

Define  $I^0(R) := I^+(R) \cap I^-(R)$  and one checks that

$$I^0(R) := \{\phi \in E(R) \mid \phi_{ij} = 0 \text{ for all but finitely many } (i, j)\}.$$

There is a trace morphism

$$(1.2) \quad \text{tr} : I^0(R) \rightarrow R; \quad \text{tr} \phi := \sum_{i \in \mathbf{Z}} \phi_{ii},$$

the sum is obviously finite. One easily verifies that  $\text{tr}[\phi, \phi'] = \sum_{i,j \in \mathbf{Z}} [\phi_{ij}, \phi'_{ji}]$  and thus  $\text{tr}[I^0(R), E(R)] \subseteq [R, R]$ . More generally, if  $R' \subseteq R$  is a subalgebra,

$$\text{tr}[I^0(R'), E(R)] \subseteq [R', R].$$

We note that this trace does not necessarily vanish on commutators. Moreover, every  $\phi \in E(R)$  can be written as  $\phi = \phi^+ + \phi^-$  with  $\phi_{ij}^+ := \delta_{i \geq 0} \phi_{ij}$  (for this  $R$  need not be unital, use  $\phi_{ij}$  for  $i \geq 0$  and 0 otherwise) and  $\phi^- = \phi - \phi^+$ . One checks that  $\phi^\pm \in I^\pm(R)$ . It follows that  $I^+(R) + I^-(R) = E(R)$ .

Finally, let  $M$  be an  $R$ -bimodule (over  $k$ , i.e. a left- $(A \otimes_k A^{op})$ -module;  $R$ -bimodules form an abelian category). Analogously to  $E(R)$ , define

$$(1.3) \quad E(M) := \{\phi = (\phi_{ij})_{i,j \in \mathbf{Z}}, \phi_{ij} \in M \mid \exists K_\phi : |i - j| > K_\phi \Rightarrow \phi_{ij} = 0\}.$$

Again using the matrix multiplication formula,  $E(M)$  is an  $E(R)$ -bimodule. If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence of  $R$ -bimodules,  $0 \rightarrow E(M') \rightarrow E(M) \rightarrow E(M'') \rightarrow 0$  is an exact sequence of  $E(R)$ -bimodules. Note that for an

ideal  $I \subseteq R$  the object  $E(I)$  is well-defined, regardless whether we regard  $I$  as an associative ring as in eq. 1.1 or an  $R$ -bimodule as in eq. 1.3.

Now let  $V$  be a finite-dimensional  $k$ -vector space and  $R_0$  an arbitrary unital subalgebra of  $\text{End}_k(V)$ . Define  $R_i := E(R_{i-1})$  for  $i = 1, \dots, n$ . Note that via  $k \rightarrow R_0$ ,  $\alpha \mapsto \alpha \cdot \mathbf{1}_{\text{End}_k(V)}$ ,  $k$  is embedded into the center of  $R_i$ . Then  $R_n = (E \circ \dots \circ E)(R_0)$  is a unital associative  $k$ -algebra. Its elements may be indexed  $\phi = (\phi_{(i_n, j_n), \dots, (i_1, j_1)}) \in \mathbb{Z}^{2n} \in R_0$ . By the properties discussed above,

$$I_i^\pm := (E \cdots_n E \circ I_i^\pm \circ E \cdots_1 E)(R_0) \quad (I^\pm \text{ in the } i\text{-th place}),$$

is an ideal in  $R_n$  (we use centered subscripts only to emphasize the numbering). Moreover,

$$\begin{aligned} I_i^+ + I_i^- &= (E \cdots E \circ I^+ \circ E \cdots E)(R_0) + (E \cdots E \circ I^- \circ E \cdots E)(R_0) \\ &= (E \cdots E \circ E \circ E \cdots E)(R_0) = R_n. \end{aligned}$$

By composing the traces of eq. 1.2 we arrive at a  $k$ -linear map  $\tau$ ,

$$\begin{aligned} \tau : I_{\text{tr}} &= I_1^0 \cap \dots \cap I_n^0 = (I^0 \circ \dots \circ I^0)(R_0) \\ &\xrightarrow{\text{tr}} \dots \xrightarrow{\text{tr}} I^0(I^0(R_0)) \xrightarrow{\text{tr}} I^0(R_0) \xrightarrow{\text{tr}} R_0 \xrightarrow{\text{Tr}} k, \end{aligned}$$

where “Tr” (as opposed to “tr”) denotes the ordinary matrix trace of  $\text{End}_k(V)$  ( $\supseteq R_0$ ). Here we have used that  $V$  is finite-dimensional over  $k$ . Using  $\text{tr}[I^0(R'), E(R)] \subseteq [R', R]$  (for subalgebras  $R' \subseteq R$ ) inductively, one sees that

$$\begin{aligned} \tau[I_{\text{tr}}, R_n] &= \text{Tr}(\text{tr} \circ \dots \circ \text{tr} \circ \text{tr})[I^0(I^0(\dots)), E(E(\dots))] \\ &\subseteq \text{Tr}(\text{tr} \circ \dots \circ \text{tr})[I^0(\dots), E(\dots)] \subseteq \text{Tr}[R_0, R_0] = 0 \end{aligned}$$

since the ordinary trace  $\text{Tr}$  vanishes on commutators. Hence,  $\tau$  factors to a morphism  $\tau : I_{\text{tr}, \text{Lie}}/[I_{\text{tr}, \text{Lie}}, R_{\text{Lie}}] \rightarrow k$ . Summarizing, for every  $n \geq 1$ , every finite-dimensional  $k$ -vector space  $V$  and every unital subalgebra  $R_0 \subseteq \text{End}_k(V)$ ,  $(R_n, (I_i^\pm), \tau)$  is a cubically decomposed algebra.

Finally, note that for any associative algebra  $R$ ,  $E(R)$  is a right- $R$ -submodule of *right- $R$ -module* endomorphisms  $\text{End}_R(R[t, t^{-1}])$  of  $R[t, t^{-1}]$ . Write elements as  $a = \sum_{i \in \mathbb{Z}} a_i t^i$ , also denoted  $a = (a_i)_i$  with  $a_i \in R$ , and let  $\phi = (\phi_{ij})$  act by  $(\phi \cdot a)_i := \sum_k \phi_{ik} a_k$ . Moreover, each  $a \in R[t, t^{-1}]$  determines a right- $R$ -module endomorphism via the multiplication operator  $x \mapsto a \cdot x$ . We find

$$R[t, t^{-1}] \hookrightarrow E(R) \hookrightarrow \text{End}_R(R[t, t^{-1}]).$$

Multiplication with  $t^i$  is represented by a matrix with a diagonal  $\dots, 1, 1, 1, \dots$ , shifted by  $i$  off the principal diagonal. Inductively,

$$(1.4) \quad R_0[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \hookrightarrow R_n \hookrightarrow \text{End}_{R_0}(R_0[t_1^{\pm 1}, \dots, t_n^{\pm 1}]).$$

See for example [11, Lec. 4] for more information regarding the case  $n = 1$  and [7, §3] for a similar procedure when  $n = 2$ .

## 2. MODIFIED CHEVALLEY-EILENBERG COMPLEXES

Suppose  $k$  is a field and  $\mathfrak{g}$  a Lie algebra over  $k$ . Let  $\mathfrak{j} \subseteq \mathfrak{g}$  be a Lie ideal. We write  $\mathfrak{j} \wedge \bigwedge^{i-1} \mathfrak{g}$  to denote the left- $U\mathfrak{g}$ -submodule of  $\mathfrak{g} \wedge \bigwedge^{i-1} \mathfrak{g} = \bigwedge^i \mathfrak{g}$  generated by elements  $j \wedge f_1 \wedge \dots \wedge f_{i-1}$  such that  $j \in \mathfrak{j}$  and  $f_1, \dots, f_{i-1} \in \mathfrak{g}$ . If  $\mathfrak{j}_i$ ,  $i = 1, 2, \dots$ , are Lie ideals, we denote by  $(\bigoplus_i \mathfrak{j}_i) \wedge \bigwedge^{i-1} \mathfrak{g}$  the module  $\bigoplus_i (\mathfrak{j}_i \wedge \bigwedge^{i-1} \mathfrak{g})$ .

*Example:* If  $k\langle s, t, u \rangle$  and  $k\langle s \rangle$  denote a 3-dimensional abelian Lie algebra along with a 1-dimensional Lie ideal, then  $\bigwedge^2 k\langle s, t, u \rangle$  is 3-dimensional with basis  $s \wedge t$ ,  $s \wedge u$  and  $t \wedge u$ . Then  $k\langle s \rangle \wedge k\langle s, t, u \rangle$  is 2-dimensional with basis  $s \wedge t$ ,  $s \wedge u$ .

The  $k$ -vector spaces  $CE(\mathfrak{j})_i := \mathfrak{j} \wedge \bigwedge^{i-1} \mathfrak{g}$  (for  $i \geq 1$ ) and  $CE(\mathfrak{j})_0 := k$  define a complex: The differential is given (for  $r \geq 1$ ) by

$$(2.1) \quad \begin{aligned} \delta &:= \delta^{[1]} + \delta^{[2]} : CE(\mathfrak{j})_{r+1} \rightarrow CE(\mathfrak{j})_r \\ \delta^{[1]}(u \wedge f_1 \wedge \dots \wedge f_r) &:= \sum_{i=1}^r (-1)^{i+1} [f_i, u] \wedge f_1 \wedge \dots \wedge \widehat{f_i} \wedge \dots \wedge f_r \\ \delta^{[2]}(u \wedge f_1 \wedge \dots \wedge f_r) &:= \sum_{i < j} (-1)^{i+j} u \wedge [f_i, f_j] \wedge f_1 \wedge \dots \wedge \widehat{f_i} \wedge \dots \wedge \widehat{f_j} \wedge \dots \wedge f_r \end{aligned}$$

(and for  $r = 0$  it is the zero map). This complex is modelled after the Chevalley-Eilenberg complex (and closely related to it, see below), its definition is taken from [2]. Note that the differential is well-defined; this uses that  $\mathfrak{j}$  is a Lie ideal. Note also that one may need to rearrange the terms to another presentation to make the first wedge factor lie in  $\mathfrak{j}$ .

*Beware:* With this definition the objects  $CE(\mathfrak{j})_i$  are neither  $U\mathfrak{g}$ -modules nor does  $U\mathfrak{g} \otimes_k CE(\mathfrak{g})_i = U\mathfrak{g} \otimes_k \bigwedge^i \mathfrak{g}$  give the conventional Chevalley-Eilenberg complex. Regarding the latter note that  $\text{id} \otimes \delta$  does *not* induce the correct differentials.

However, we have

$$k \otimes_{U\mathfrak{g}} \left( U\mathfrak{g} \otimes_k \bigwedge^i \mathfrak{g} \right) \cong \bigwedge^i \mathfrak{g} = CE(\mathfrak{g})_i,$$

where the expression in parentheses is an object in the complex (of free left- $U\mathfrak{g}$ -modules) giving the conventional Chevalley-Eilenberg resolution. In this special case the differentials of the usual Chevalley-Eilenberg complex and the one on the right-hand side agree. This leads to the following fact:

**Lemma 1** ([2, Lemma 1(a)]).  *$CE(\mathfrak{g})_\bullet$  is a complex of  $k$ -vector spaces and is quasi-isomorphic to  $k \otimes_{U\mathfrak{g}}^L k$ . In particular*

$$H_i(\mathfrak{g}, k) = H_i(CE(\mathfrak{g})_\bullet, k) \text{ and } H^i(\mathfrak{g}, k) = H^i(\text{Hom}_k(CE(\mathfrak{g})_\bullet, k)).$$

*Proof.* We use that the conventional Chevalley-Eilenberg complex  $U\mathfrak{g} \otimes_k \bigwedge^\bullet \mathfrak{g}$  is a projective (and thus flat) resolution of  $k$  (placed in deg. 0) in  $\mathbf{D}^-_{U\mathfrak{g}} \text{Mod}$ .

$$H_i(\mathfrak{g}, k) := H_i(k \otimes_{U\mathfrak{g}}^L k) \cong H_i(k \otimes_{U\mathfrak{g}} (U\mathfrak{g} \otimes_k \bigwedge^\bullet \mathfrak{g})) \cong H_i(CE(\mathfrak{g})_\bullet),$$

where we have used that the differential of eq. 2.1 agrees with the one induced via  $\text{id}_k \otimes \delta$  of the conventional Chevalley-Eilenberg complex. Similarly for  $H^i(\mathfrak{g}, k)$ .  $\square$

*Example:*  $H_0(CE(\mathfrak{g})_\bullet) = k$ ,  $H_1(CE(\mathfrak{g})_\bullet) = H_1(\mathfrak{g}, k) = \mathfrak{g}^{\text{ab}}$ ,  $H_1(CE(\mathfrak{j})_\bullet) = \mathfrak{j}/[\mathfrak{j}, \mathfrak{g}] = H_0(\mathfrak{g}, \mathfrak{j})$ .

*Beware:* The complex  $CE(\mathfrak{j})_\bullet$  should not be confused with  $\mathfrak{j} \otimes_{U\mathfrak{g}} (U\mathfrak{g} \otimes_k \bigwedge^{\bullet-1} \mathfrak{g})$ , where the latter is the (shifted) conventional Chevalley-Eilenberg complex. For example in degree 2 we get  $\mathfrak{j} \otimes_k \mathfrak{g}$ , while  $CE(\mathfrak{j})_2 = \mathfrak{j} \wedge \mathfrak{g}$ . Taking the abelian Lie algebras  $\mathfrak{g} := k\langle s, t, u \rangle$  and  $\mathfrak{j} := k\langle s \rangle$  as above, all differentials of either complex vanish (so the homology agrees with the entries of the complex), but  $\dim(\mathfrak{j} \otimes_k \mathfrak{g}) = 3$  and  $\dim(\mathfrak{j} \wedge \mathfrak{g}) = 2$ .

**Lemma 2.** *Suppose we are given an exact sequence*

$$N_\bullet = [N_{n+1} \rightarrow N_n \rightarrow \dots \rightarrow N_0]_{n+1,0}$$

*in  $\mathbf{K}^-_{(U\mathfrak{g})\text{Mod}}$ , where the  $N_i$  are Lie ideals of  $\mathfrak{g}$  (or finite direct sums thereof).*

- (1) *There is a second quadrant homological spectral sequence  $(E_{p,q}^r, d_r)$  converging to zero such that*

$$E_{p,q}^1 = H_q(CE(N_p)_\bullet). \quad (d_r : E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r)$$

- (2) *There is a first quadrant cohomological spectral sequence  $(E_r^{p,q}, d^r)$  converging to zero such that*

$$E_1^{p,q} = H^q(\text{Hom}_k(CE(N_p)_\bullet, k)). \quad (d^r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1})$$

- (3) *The following differentials are isomorphisms:*

$$d_{n+1} : E_{n+1,1}^{n+1} \rightarrow E_{0,n+1}^{n+1} \quad \text{and} \quad d^{n+1} : E_{n+1}^{0,n+1} \rightarrow E_{n+1}^{n+1,1}.$$

- (4) *Suppose  $H_p : N_p \rightarrow N_{p+1}$  is a contracting homotopy for  $N_\bullet$ . Then*

$$(d_{n+1})^{-1} = H_n \delta_1 H_{n-1} \cdots \delta_{n-1} H_1 \delta_n H_0 = H_n \prod_{i=1, \dots, n} (\delta_i H_{n-i})$$

(where the last product depends on the ordering and refers to composition),  
and

$$(d^{n+1})^{-1} = H_0^* \delta_n^* H_1^* \cdots \delta_1^* H_n^* = H_0^* \prod_{i=n, \dots, 1} (\delta_i^* H_{n+1-i}^*),$$

where we write  $f^* = \text{Hom}_{U\mathfrak{g}}(f, k)$  as a short-hand.

*Proof.* Parts (1)-(3) are [2, Lemma 1(a)]. More precisely, for (1) use the bicomplex spectral sequence for

$$E_{p,q}^0 = CE(N_p)_q \quad \text{and} \quad E_0^{p,q} = \text{Hom}_k(CE(N_p)_q, k).$$

If we take differentials ‘ $\rightarrow$ ’ for forming the  $E^0$ -page, the  $E^1$ -page vanishes since  $N_\bullet$  is exact and so are the rows  $CE(N_\bullet)_i$  for constant  $i$ , so  $E^\infty = E^1 = 0$ . Then use the bicomplex spectral sequences with differential ‘ $\downarrow$ ’ on the  $E^0$ -page for our claim. It also converges to zero then; (2) is analogous. (3) The bicomplex is horizontally supported in  $[0, n+1]$ . (4) Diagram chase.  $\square$

### 3. THE CUBE COMPLEX

Let  $(A, (I_i^\pm), \tau)$  be an  $n$ -fold cubically decomposed algebra over a field  $k$ , see Def. 1, i.e. we are given the following datum:

- an associative unital (not necessarily commutative)  $k$ -algebra  $A$ ;
- two-sided ideals  $I_i^+, I_i^-$  such that  $I_i^+ + I_i^- = A$  for  $i = 1, \dots, n$ ;
- writing  $I_i^0 := I_i^+ \cap I_i^-$  and  $I_{\text{tr}} := I_1^0 \cap \cdots \cap I_n^0$ , a  $k$ -linear map

$$\tau : I_{\text{tr}, \text{Lie}} / [I_{\text{tr}, \text{Lie}}, A_{\text{Lie}}] \rightarrow k.$$

See §1 to see how this type of structure arises. As a short-hand, define  $\mathfrak{g} := A_{\text{Lie}}$ . Next, we shall construct a certain complex of  $\mathfrak{g}$ -modules, strongly inspired by a cubical object appearing in [2] (we prefer to transform the cubical object into a complex however).

For any elements  $s_1, \dots, s_n \in \{+, -, 0\}$  we define the *degree*  $\deg(s_1, \dots, s_n) := 1 + \#\{i \mid s_i = 0\}$ . We define a  $\mathfrak{g}$ -module  $N_0 := \mathfrak{g}$  and for  $p \geq 1$

$$(3.1) \quad N_p := \coprod_{s_1, \dots, s_n \in \{+, -, 0\}} I_1^{s_1} \cap I_2^{s_2} \cap \cdots \cap I_n^{s_n} \quad (\text{with } \deg(s_1, \dots, s_n) = p).$$

We shall denote the components  $f = (f_{s_1 \dots s_n})$  of elements in  $N_p$  with indices in terms of  $s_1, \dots, s_n \in \{+, -, 0\}$ . Clearly  $N_p = 0$  for  $p > n+1$ . We shall treat all  $N_p$  as  $\mathfrak{g}$ -modules.



*Example:* For  $n = 1$  we have

$$N_2 = I_1^0, \quad N_1 = I_1^+ \oplus I_1^-$$

and elements would be denoted  $f = (f_0) \in N_2$  and  $g = (g_+, g_-) \in N_1$ . For  $n = 2$  we have

$$\begin{aligned} N_3 &= I_1^0 \cap I_2^0, & N_2 &= \left( \coprod_{s_1 \in \{+, -\}} I_1^{s_1} \cap I_2^0 \right) \oplus \left( \coprod_{s_2 \in \{+, -\}} I_1^0 \cap I_2^{s_2} \right) \\ N_1 &= \coprod_{s_1, s_2 \in \{+, -\}} I_1^{s_1} \cap I_2^{s_2}. \end{aligned}$$

**Definition 2.** Define morphisms of  $\mathfrak{g}$ -modules  $\partial_i : N_i \rightarrow N_{i-1}$  by

$$(\partial_i f)_{s_1 \dots s_n} := \begin{cases} \sum_{\{l | s_l = +, -\}} (-1)^{\#\{j | j > l \text{ and } s_j = 0\}} f_{s_1 \dots \underset{(l)}{0} \dots s_n} & (\text{for } i \geq 2) \\ \sum_{s_1 \dots s_n \in \{+, -\}} (-1)^{s_1 + \dots + s_n} f_{s_1 \dots s_n} & (\text{for } i = 1), \end{cases}$$

where  $(-1)^\pm = \pm 1$  and

- (1)  $\{l | s_l = +, -\}$  denotes the set of all  $l = 1, \dots, n$  such that  $s_l \in \{+, -\}$ ,
- (2)  $f_{s_1 \dots \underset{(l)}{0} \dots s_n}$  refers to plugging in 0 for the  $l$ -th index.

As the  $N_i$  are direct sums of ideals in  $\mathfrak{g}$ , it is clear that the  $\partial_i$  are indeed  $\mathfrak{g}$ -module homomorphisms. For example, if  $n = 1$  we obtain

$$(\partial_2 f)_\pm = f_0 \quad \text{and} \quad \partial_1 g = g_+ - g_-$$

for  $f \in N_2$  and  $g \in N_1$ . For  $n = 2$  and  $s_1, s_2 \in \{+, -\}$  we obtain

$$\begin{aligned} (\partial_2 g)_{s_1 s_2} &= f_{0 s_2} + f_{s_1 0} & \text{and} & & (\partial_3 f)_{\pm 0} &= -f_{00} \\ (\partial_1 h) &= f_{++} - f_{+-} - f_{-+} + f_{--} & & & (\partial_3 f)_{0 \pm} &= f_{00} \end{aligned}$$

for  $f \in N_3$ ,  $g \in N_2$  and  $h \in N_1$ .

**Definition 3.** Let  $(A, (I_i^\pm), \tau)$  be an  $n$ -fold cubically decomposed algebra over a field  $k$ . A system of good idempotents are elements  $P_i^+ \in A$  for  $i = 1, \dots, n$  such that for all  $i$ :

- (1)  $P_i^{+2} = P_i^+$ .
- (2)  $P_i^+ A \subseteq I_i^+$ .
- (3)  $P_i^- A \subseteq I_i^-$  (where we define  $P_i^- := \mathbf{1}_A - P_i^+$ ).

We note that the  $P_i^-$  are also idempotents and  $P_i^+ + P_i^- = \mathbf{1}_A$ .

**Definition 4.** Fix a system of good idempotents  $P_1^\pm, \dots, P_n^\pm$ . Define  $k$ -vector space morphisms  $H_i : N_i \rightarrow N_{i+1}$  for  $i = 0$  by

$$(3.2) \quad (H_0 f)_{s_1 \dots s_n} := (-1)^{s_1 + \dots + s_n} P_1^{s_1} \dots P_n^{s_n} f$$

and for  $1 \leq i \leq n$  by

$$(3.3) \quad (H_i f)_{s_1 \dots s_n} := (-1)^{s_1 + \dots + s_b} P_1^{s_1} \dots P_b^{s_b} \sum_{\gamma_1, \dots, \gamma_{b+1} \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_b} P_{b+1}^{-\gamma_{b+1}} f_{\gamma_1 \dots \gamma_{b+1} s_{b+2} \dots s_n},$$

where  $b$  denotes the largest index such that  $s_1, s_2, \dots, s_b \in \{\pm\}$  or  $b = 0$  if none (and so  $s_{b+1} = 0$  if  $b < n$ ;  $b+1$  is the index of the “leftmost zero”). The expression  $P_{b+1}^{-\gamma_{b+1}}$  means  $P_{b+1}^-$  for  $\gamma_{b+1} = +$  and  $P_{b+1}^+$  for  $\gamma_{b+1} = -$ .

*Example:* For  $n = 1$  and  $s_1 \in \{+, -\}$  we obtain

$$(H_0 f)_{s_1} = (-1)^{s_1} P_1^{s_1} f \quad \text{and} \quad (H_1 g)_0 = P_1^- g_+ + P_1^+ g_-$$

for  $f \in N_0$  and  $g \in N_1$ . For  $n = 2$  and  $s_1, s_2 \in \{+, -\}$  we obtain

$$(3.4) \quad \begin{aligned} (H_0 f)_{s_1 s_2} &= (-1)^{s_1 + s_2} P_1^{s_1} P_2^{s_2} f \\ (H_1 g)_{s_1 0} &= (-1)^{s_1} P_1^{s_1} \sum_{\gamma_1, \gamma_2} (-1)^{\gamma_1} P_2^{-\gamma_2} f_{\gamma_1 \gamma_2} \\ (H_1 g)_{0 s_2} &= P_1^- g_{+ s_2} + P_1^+ g_{- s_2} \\ (H_2 h)_{00} &= P_1^- g_{+0} + P_1^+ g_{-0} \end{aligned}$$

for  $f \in N_0$ ,  $g \in N_1$  and  $h \in N_2$ . For example,

$$(H_1 g)_{+0} = P_1^+ (P_2^- (f_{++} - f_{-+}) + P_2^+ (f_{+-} - f_{--})).$$

**Lemma 3.** *Equipped with these morphisms*

$$(3.5) \quad N_\bullet = [N_{n+1} \xrightleftharpoons[H_n]{\partial_{n+1}} N_n \xrightleftharpoons[H_{n-1}]{\partial_n} \cdots \xrightleftharpoons[H_0]{\partial_1} N_0]_{n+1,0}$$

is a complex of  $\mathfrak{g}$ -modules with differentials  $\partial_\bullet$  and contracting homotopies  $H_\bullet$  (in the category of  $k$ -vector spaces), i.e.

$$\partial_{i-1} \circ \partial_i = 0 \quad H_{i-1} \circ \partial_i + \partial_{i+1} \circ H_i = \text{id}_{N_i} \quad \text{for all } i.$$

This complex is modelled after a cubical object used by Beilinson [2]. In view of our needs for explicit computations and the construction of a contracting homotopy, our presentation is quite different. The contracting homotopy is new.

*Proof.* This is entirely a combinatorial verification. We leave a detailed proof to the reader. One possible approach is to construct the complex inductively: assume the dimension  $n$  is fixed and then proceed inductively along  $m = 1, 2, \dots, n$ . We shall usually use a left superscript - as  $m$  in  ${}^m \partial_i$  or  ${}^m H_0$  - to denote the induction

variable  $m$ . For  $m = 1$  check by hand that  $[I_1^0 \xrightleftharpoons[{}^1 H_1]{{}^1 \partial_2} I_1^+ \oplus I_1^+ \xrightleftharpoons[{}^1 H_0]{{}^1 \partial_1} \mathcal{E}_V]_{2,0}$  is exact

(indexed in degrees  $2, 1, 0$ ). Denote this complex by  $[{}^1 M_2 \xrightleftharpoons[{}^1 H_1]{{}^1 \partial_2} {}^1 M_1 \xrightleftharpoons[{}^1 H_0]{{}^1 \partial_1} {}^1 M_0]_{2,0}$ .

Then for each  $m = 2, 3, \dots$  proceed by induction and observe that given

$$(3.6) \quad [{}^m M_{m+1} \xrightleftharpoons[{}^m H_m]{{}^m \partial_{m+1}} {}^m M_m \rightleftharpoons \cdots \rightleftharpoons {}^m M_1 \xrightleftharpoons[{}^m H_0]{{}^m \partial_1} {}^m M_0]_{m+1,0}$$

the total complex of the 2-row bicomplex

$$(3.7) \quad \begin{array}{ccccc} ({}^m M_{m+1} \cap I_{m+1}^0) & \rightleftharpoons & \cdots & \rightleftharpoons & ({}^m M_1 \cap I_{m+1}^0) \\ \downarrow & & & & \downarrow \\ ({}^m M_{m+1} \cap I_{m+1}^+) & \rightleftharpoons & \cdots & \rightleftharpoons & ({}^m M_1 \cap I_{m+1}^+) \\ \oplus ({}^m M_{m+1} \cap I_{m+1}^-) & \rightleftharpoons & \cdots & \rightleftharpoons & \oplus ({}^m M_1 \cap I_{m+1}^-) \rightleftharpoons {}^m M_0, \end{array}$$

(whose columns in horizontal degrees  $m+1, \dots, 1$  are formed from objects  $*$  in the shape

$$0 \longrightarrow * \cap I_{m+1}^0 \longrightarrow (* \cap I_{m+1}^+) \oplus (* \cap I_{m+1}^-) \longrightarrow 0$$

and the column at 0 remains just  $*$ ) yields just a complex as in eq. 3.6, but for the  $(m+1)$ -th induction step. The differential of the total complex is given by  $D := d \pm {}^m \partial_*$ , where the downward differential  $d$  denotes the morphism  $f_{*,+} := f_{*,0}$ ,  $f_{*,+} := f_{*,0}$ , where  $*$  denotes the indexing of the previous induction step.

By induction this yields exactly the differentials as we claim, and formula for the contracting homotopy can be checked analogously by induction.  $\square$

We combine Lemma 3 with Lemma 2: The fact that the bicomplexes of Prop. 2 are supported horizontally in  $[n+1, 0]$  (homologically, i.e. for  $E_{\bullet, \bullet}^{\bullet}$ ) and  $[0, n+1]$  respectively (cohomologically, i.e. for  $E_{\bullet, \bullet}^{\bullet}$ ) implies that we have edge morphisms

$$\begin{aligned} \rho_1 : E_{n+1,1}^{n+1} &\rightarrow E_{n+1,1}^1 & \text{and} & & \rho_2 : E_{0,n+1}^1 &\rightarrow E_{0,n+1}^{n+1} \\ \wp_1 : E_{n+1}^{0,n+1} &\rightarrow E_1^{0,n+1} & \text{and} & & \wp_2 : E_1^{n+1,1} &\rightarrow E_{n+1}^{n+1,1}. \end{aligned}$$

Next, we identify the involved objects: Using Lemma 1 and  $N_0 = \mathfrak{g}$  (by definition of  $N_0$ ), we compute

$$\begin{aligned} E_{0,n+1}^1 &= H_{n+1}(CE(N_0)\bullet) = H_{n+1}(CE(\mathfrak{g})\bullet) \cong H_{n+1}(\mathfrak{g}, k) \\ E_{n+1,1}^1 &= H_1(CE(N_{n+1})\bullet) = N_{n+1}/[\mathfrak{g}, N_{n+1}] \cong I_{\text{tr}}/[I_{\text{tr}}, \mathfrak{g}] \\ E_1^{n+1,1} &= H^1(\text{Hom}_k(CE(N_{n+1})\bullet, k)) = \text{Hom}_k(I_{\text{tr}}/[I_{\text{tr}}, \mathfrak{g}], k) \\ E_1^{0,n+1} &= H^{n+1}(\text{Hom}_k(CE(N_0)\bullet, k)) \cong H^{n+1}(\mathfrak{g}, k). \end{aligned}$$

**Definition 5** ([2]). *Let  $(A, (I_i^\pm), \tau)$  be an  $n$ -fold cubically decomposed algebra over a field  $k$  and  $\mathfrak{g} := A_{\text{Lie}}$  its Lie algebra. Define*

$$\text{res}_* : H_{n+1}(\mathfrak{g}, k) \rightarrow k \quad \text{res}_* := \tau \circ \rho_1 \circ (d_{n+1})^{-1} \circ \rho_2$$

and

$$\text{res}^* : k \rightarrow H^{n+1}(\mathfrak{g}, k) \quad \text{res}^*(1) := (\wp_1 \circ (d^{n+1})^{-1} \circ \wp_2)\tau,$$

where for  $\text{res}^*$  we read  $\tau$  as an element of  $E_1^{n+1,1}$ . We will call  $\phi := \text{res}^*(1)$  the Tate extension class.

**Remark 1.** *It follows from the construction of  $\text{res}_*$ ,  $\text{res}^*$  that*

$$(3.8) \quad \text{res}^*(\alpha)(X_0 \wedge \dots \wedge X_n) = \alpha \text{res}_* X_0 \wedge \dots \wedge X_n.$$

#### 4. CONCRETE FORMALISM

Let  $(A, (I_i^\pm), \tau)$  be an  $n$ -fold cubically decomposed algebra over a field  $k$ . In §3 we have constructed a canonical morphism

$$\text{res}_* : H_{n+1}(\mathfrak{g}, k) \rightarrow k,$$

where  $\mathfrak{g} := A_{\text{Lie}}$  is the Lie algebra associated to  $A$ . In this section we will obtain an explicit formula for this morphism.

Given the definition of  $\text{res}_*$ , i.e. Def. 5, Lemma 2 tells us that it can be given explicitly in terms of differentials of the Chevalley-Eilenberg complexes  $CE(-)\bullet$  (as developed in §2) and contracting homotopies of the cube complex  $N_\bullet$  (as developed in §3), namely

$$(4.1) \quad \text{res}_* = \tau \circ \rho_1 \circ (d_{n+1})^{-1} \circ \rho_2 = \tau \circ \rho_1 H_n \prod_{i=1, \dots, n} (\delta_i H_{n-i}) \rho_2.$$

The contracting homotopies  $H_0, H_1, \dots$  depend on the choice of a good system of idempotents, see Def. 3. Different choices will yield formulas that may look different, but as  $\text{res}_*$  was defined entirely independently of the choice of any idempotents, all such formulas actually must agree.

Suppose a representative  $\theta := f_0 \wedge \dots \wedge f_n$  with  $f_0, \dots, f_n \in N_0$  is given (note that  $N_0$  equals  $\mathfrak{g}$  as a left- $U\mathfrak{g}$ -module by definition, so it is valid to treat all  $f_i$  on

equal footing). We shall compute  $\text{res}_* \theta$  in several steps, starting with  $\theta_{0,n} := \rho_2 \theta$ , then following

$$(4.2) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & | & & \\ & & \theta_{1,n} & \xleftarrow{H_0} & \theta_{0,n} & n & \\ & & \vdots & & & \vdots & \\ & \theta_{n,1} & \xleftarrow{H_{n-1}} & \theta_{n-1,1} & & 1 & \begin{array}{c} q \\ \uparrow \\ + \end{array} \\ & \downarrow & & & & & \\ \theta_{n+1,0} & \xleftarrow{H_n} & \theta_{n,0} & & & 0 & \\ \hline n+1 & & n & & n-1 & \dots & 0 \end{array}$$

as prescribed by eq. 4.1. This graphical arrangement elucidates the position of the term of each step in the computation in the spectral sequence from which eq. 4.1 originates, see Lemma 2. However, for us each  $\theta_{*,*}$  will be an  $E^0$ -page representative of the respective  $E^*$ -page term. Finally  $\text{res}_* \theta = \tau \rho_1 \theta_{n+1,0}$ . We note that  $\rho_1, \rho_2$  are just edge maps, i.e. an inclusion of a subobject and a quotient surjection. Hence, as we work with explicit representatives anyway, the operation of these maps is essentially invisible (e.g. in the quotient case it just means that our representative generates a larger equivalence class).

We will need a convenient notation for elements of this complex.<sup>1</sup>

(Notation A) We will write  $\theta_{p,q-p|s_1 \dots s_n}^{w_1 \dots w_p} \in N_p$  for the summands in any expression of the shape

$$(4.3) \quad \theta_{p,q-p} = \sum_{\substack{w_1 \dots w_p \\ \in \{1, \dots, n\}}} \sum_{s_1 \dots s_n} \theta_{p,q-p|s_1 \dots s_n}^{w_1 \dots w_p} \wedge f_1 \wedge \dots \wedge \widehat{f_{w_1}} \wedge \dots \wedge \widehat{f_{w_p}} \wedge \dots \wedge f_n,$$

where

- $(p, q-p)$  denotes the location of the element in the bicomplex as in fig. 4.2,
- $s_1, \dots, s_n \in \{0, +, -\}$  denotes the component (= direct summand) of  $N_p$  as in eq. 3.1,  $f_1, \dots, f_n \in \mathfrak{g}$ ,
- the additional superscripts  $w_1, \dots, w_p \in \{1, \dots, n\}$  are used to indicate the omission of wedge factors.

Note that the values  $\theta_{p,q|s_1 \dots s_n}^{w_1 \dots w_p}$  are not necessarily uniquely determined since the individual wedge tails need not be linearly independent.

(Notation B) We also need a short-hand for the summands in any expression of the shape

$$(4.4) \quad \theta_{p,q-p-1} = \sum_{\substack{w_1 \dots w_p, w_a, w_b \\ \in \{1, \dots, n\}}} \sum_{s_1 \dots s_n} \theta_{p,q|s_1 \dots s_n}^{w_1 \dots w_p} \parallel_{w_a, w_b} \\ \wedge [f_{w_a}, f_{w_b}] \wedge f_1 \wedge \dots \wedge \widehat{f_{w_1}} \dots \widehat{f_{w_a}} \dots \widehat{f_{w_b}} \dots \widehat{f_{w_p}} \dots \wedge f_n.$$

Again  $s_1, \dots, s_n$  denotes the component in  $N_p$ ,  $w_1, \dots, w_p$  omitted wedge factors. Moreover,  $w_a$  and  $w_b$  denote two additional omitted wedge factors and simultaneously indicate that  $[f_{w_a}, f_{w_b}]$  appears as an additional wedge factor. As for the

<sup>1</sup>Indeed 99% of the computation lies in the choice of a manageable notation.

previous notation, the elements  $\theta_{p,q|s_1\dots s_n}^{w_1\dots w_p||w_a,w_b} \in N_p$  are not uniquely determined. We will explain how these expressions arise soon.

*Combinatorial Preparation:* We define for arbitrary  $1 \leq p \leq n$  and  $w_1, \dots, w_p \in \{1, \dots, n\}$  the ‘sign function’ (a generalization of the signum of a permutation)

$$(4.5) \quad \rho(w_1, \dots, w_p) := (-1)^{\sum_{k=1}^p \sum_{j < k} \delta_{w_j < w_k}}.$$

By abuse of language we do not carry the value  $p$  in the notation for  $\rho$  as it will always be clear from the number of arguments which variant is used. It is easy to see that  $\rho(w_1) = +1$  and  $\rho(w_1, w_2) = (-1)^{\delta_{w_1 < w_2}}$ . For  $p = n$  we have

$$(4.6) \quad \rho(w_1, \dots, w_n) = \text{sgn} \begin{pmatrix} 1 & \cdots & n \\ w_1 & \cdots & w_n \end{pmatrix}.$$

We shall need the inductive formula (which is easy to check by induction)

$$(4.7) \quad (-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}} \rho(w_1, \dots, w_p) = \rho(w_1, \dots, w_{p+1}).$$

**Proposition 1.** *Suppose  $\theta := f_0 \wedge \dots \wedge f_n$  with  $f_i \in N_0 = \mathfrak{g}$ . Moreover, suppose  $P_1^+, \dots, P_n^+$  is a good system of idempotents as in Def. 3. Assume either that*

- $P_1^+, \dots, P_n^+$  are pairwise commuting, or
- $f_1, \dots, f_n$  (without  $f_0!$ ) are pairwise commuting.

*Then for every  $p \geq 0$  the element  $\theta_{p+1,q}$  is of the shape as in eq. 4.3 and for  $\gamma_1 \dots \gamma_{n-p} \in \{+, -\}$  we have*

$$\begin{aligned} \theta_{p+1,q|\gamma_1\dots\gamma_{n-p}}^{w_1\dots w_p} \underbrace{0\dots 0}_p &= (-1)^p (-1)^{w_1+\dots+w_p} \rho(w_1, \dots, w_p) \\ &\quad (-1)^{\gamma_1+\dots+\gamma_{n-p}} P_1^{\gamma_1} \dots P_{n-p}^{\gamma_{n-p}} \\ &\quad \sum_{\gamma_{n-p+1}^* \dots \gamma_n^* \in \{\pm\}} (-1)^{\gamma_{n-p+1}^*+\dots+\gamma_n^*} \\ &\quad \left( P_{n-p+1}^{(-\gamma_{n-p+1}^*)} \text{ad}(f_{w_p}) P_{n-p+1}^{\gamma_{n-p+1}^*} \right) \\ &\quad \dots \left( P_n^{(-\gamma_n^*)} \text{ad}(f_{w_1}) P_n^{\gamma_n^*} \right) f_0. \end{aligned}$$

Here  $\rho(w_1, \dots, w_p)$  is the sign function defined in eq. 4.5. For  $p = 0$  the expression  $\rho(w_1, \dots, w_p)$  and the whole sum  $(\Sigma_{\{\pm\}}(\dots))$  in  $(\Sigma_{\{\pm\}}(\dots))f_0$  should be read as  $+1$  (giving the right-hand side of eq. 4.8 below).

- Note that no terms of the shape as in eq. 4.4 appear. This is not entirely obvious in view of the definition of  $\delta^{[2]}$ , see eq. 2.1.
- The formula does not compute  $\theta_{p+1,q|s_1\dots s_n}^{w_1\dots w_p}$  for arbitrary  $s_1 \dots s_n$  of degree  $p+1$ . This is due to the fact that we only have further use for the ones treated.

*Proof.* We prove this by induction. For  $p = 0$  the claim reads

$$(4.8) \quad \theta_{1,q|\gamma_1\dots\gamma_n} = (-1)^{\gamma_1+\dots+\gamma_n} P_1^{\gamma_1} \dots P_n^{\gamma_n} f_0$$

and in view of eq. 3.2 this proves the claim in this case. Now we proceed by induction. Assume the case  $p$  is settled, i.e. in the notation of eq. 4.3  $\theta_{p+1,q|\gamma_1\dots\gamma_{n-p}}^{w_1\dots w_p} \underbrace{0\dots 0}_p$

is exactly as in our claim. Next, we need to apply the differential  $\delta_q = \delta_q^{[1]} + \delta_q^{[2]}$  of

the Chevalley-Eilenberg resolution, see eq. 2.1. The contribution of  $\delta_q^{[1]}$  will be relevant, but for  $\delta_q^{[2]}$  we shall see that (after applying the next contracting homotopy) the contribution vanishes. We treat each  $\delta^{[i]}$ ,  $i = 1, 2$  separately:

(1) Consider  $\delta_q^{[1]}$  in eq. 2.1. The sum  $\Sigma_i$  *loc. cit.* maps components indexed by  $w_1, \dots, w_p$  to components of  $\delta^{[1]}_{p,q}$ , indexed by  $w_1, \dots, w_p$  and an additional  $w_{p+1} \in \{1, \dots, n\} \setminus \{w_1, \dots, w_p\}$  – they correspond to the summands of  $\delta^{[1]}_{p,q}$  and to the additional omitted wedge factor respectively. Moreover, the formula imposes signs  $(-1)^{i+1}$ , but here  $i$  depends on the numbering of the wedges  $(\dots \wedge \dots \wedge \dots)$ . In the notation of eq. 4.3 the subscript  $j$  of  $f_j$  does not necessarily indicate the  $f_j$  sits in the  $j$ -th wedge, due to the possible omission of wedge factors  $f_{w_1}, \dots, f_{w_p}$  on the left-hand side of it. To compensate for that in the following computation the term  $(-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}}$  appears, sign-counting the omission on the left of the new-to-be-omitted  $w_{p+1}$  in the component of  $\delta^{[1]}_{p+1,q}$ . As  $p$  remains constant, the indexing  $\gamma_1 \dots \gamma_{n-p} 0 \dots 0$  remains unaffected. We get for  $(\delta^{[1]}_{p+1,q})^{w_1 \dots w_p w_{p+1}}_{p+1,q-1 | \gamma_1 \dots \gamma_{n-p} \underbrace{0 \dots 0}_p}$  the expression

$$\begin{aligned} &= (-1)^p (-1)^{w_{p+1}+1} (-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}} \text{ad}(f_{w_{p+1}}) \\ &(-1)^{w_1 + \dots + w_p} \rho(w_1, \dots, w_p) \\ &(-1)^{\gamma_1 + \dots + \gamma_{n-p}} P_1^{\gamma_1} \dots P_{n-p}^{\gamma_{n-p}} \\ &\sum_{\gamma_{n-p+1}^* \dots \gamma_n^* \in \{\pm\}} (-1)^{\gamma_{n-p+1}^* + \dots + \gamma_n^*} \\ &\left( P_{n-p+1}^{(-\gamma_{n-p+1}^*)} \text{ad}(f_{w_p}) P_{n-p+1}^{\gamma_{n-p+1}^*} \right) \dots \left( P_n^{(-\gamma_n^*)} \text{ad}(f_{w_1}) P_n^{\gamma_n^*} \right) f_0. \end{aligned}$$

Next, we need to apply the contracting homotopy  $H_{p+1}$ . Note that we have  $p+1 \geq 1$ , so eq. 3.3 applies. Note that for indices  $\gamma_1^\dagger \dots \gamma_{n-p-1}^\dagger \underbrace{0 \dots 0}_{p+1}$  with  $\gamma_1^\dagger \dots \gamma_{n-p-1}^\dagger \in \{\pm\}$  (i.e. indices of degree  $p+2$ , cf. eq. 3.1) the index  $\gamma_1^\dagger \dots \gamma_{n-p-1}^\dagger \underbrace{0 \dots 0}_p$  has degree  $p+1$ . The latter have been computed above. We obtain for

$$(H\delta^{[1]}_{p+1,q})^{w_1 \dots w_p w_{p+1}}_{p+2,q-1 | \gamma_1^\dagger \dots \gamma_{n-p-1}^\dagger \underbrace{0 \dots 0}_{p+1}}$$

the expression

$$\begin{aligned} &= (-1)^{\gamma_1^\dagger + \dots + \gamma_{n-p-1}^\dagger} P_1^{\gamma_1^\dagger} \dots P_{n-p-1}^{\gamma_{n-p-1}^\dagger} \\ &\sum_{\gamma_1, \dots, \gamma_{(n-p-1)+1} \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_{n-p-1}} P_{(n-p-1)+1}^{-(\gamma_{(n-p-1)+1})} \\ &(\delta\theta_{p+1,q})^{w_1 \dots w_{p+1}}_{p+1,q-1 | \gamma_1 \dots \gamma_{n-p} \underbrace{0 \dots 0}_p}. \end{aligned}$$

Next, we expand this using our previous computation and obtain (by noting that many signs are squares and thus +1)

$$\begin{aligned}
&= (-1)^{p+1} (-1)^{\gamma_1^\dagger + \dots + \gamma_{n-p-1}^\dagger} (-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}} \\
&(-1)^{w_1 + \dots + w_{p+1}} \rho(w_1, \dots, w_p) P_1^{\gamma_1^\dagger} \dots P_{n-p-1}^{\gamma_{n-p-1}^\dagger} \sum_{\gamma_{n-p} \in \{\pm\}} (-1)^{\gamma_{n-p}} \\
&\left( \sum_{\gamma_1, \dots, \gamma_{n-p-1} \in \{\pm\}} P_1^{\gamma_1} \dots P_{n-p-1}^{\gamma_{n-p-1}} \right) P_{n-p}^{-\gamma_{n-p}} \text{ad}(f_{w_{p+1}}) P_{n-p}^{\gamma_{n-p}} \\
&\sum_{\gamma_{n-p+1}^* \dots \gamma_n^* \in \{\pm\}} (-1)^{\gamma_{n-p+1}^* + \dots + \gamma_n^*} \\
&\left( P_{n-p+1}^{(-\gamma_{n-p+1}^*)} \text{ad}(f_{w_p}) P_{n-p+1}^{\gamma_{n-p+1}^*} \right) \dots \left( P_n^{(-\gamma_n^*)} \text{ad}(f_{w_1}) P_n^{\gamma_n^*} \right) f_0.
\end{aligned}$$

The sum in parantheses is the identity since for all  $i$  we have  $P_i^+ + P_i^- = \text{id}$  by Def. 3. Up to the naming of the indices, and after using eq. 4.7, this is exactly our claim in the case  $p+1$  (and this is true despite the fact that we have only considered  $\delta^{[1]}$  so far – because we shall next show that the contribution from  $H \circ \delta^{[2]}$  vanishes).

(2) Consider  $\delta_q^{[2]}$  in eq. 2.1. Using the notation of eq. 4.3 we may write

$$\theta_{p+1,q} = \bigoplus_{\deg(s_1 \dots s_n) = p+1} \sum_{\substack{w_1 \dots w_p \\ \in \{1, \dots, n\}, \\ \text{pairw. diff.}}} \theta_{p+1,q|s_1 \dots s_n}^{w_1 \dots w_p} \wedge f_1 \wedge \widehat{f_{w_1}} \dots \widehat{f_{w_p}} \wedge f_n$$

Therefore  $\delta^{[2]} \theta_{p+1,q}$  equals

$$\begin{aligned}
&= \bigoplus_{\deg(s_1 \dots s_n) = p+1} \sum_{\substack{w_1 \dots w_p \\ \in \{1, \dots, n\}, \\ \text{pairw. diff.}}} \sum_{\substack{w_{p+1} < w_{p+2} \\ \in \{1, \dots, n\} \setminus \{w_1, \dots, w_p\}}} (-1)^{w_{p+1} + w_{p+2}} \\
&(-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}} (-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+2}\}} \\
&\theta_{p+1,q|s_1 \dots s_n}^{w_1 \dots w_p} \wedge [f_{w_{p+1}}, f_{w_{p+2}}] \wedge f_1 \wedge \widehat{f_{w_1}} \dots \widehat{f_{w_{p+1}}} \dots \widehat{f_{w_{p+2}}} \dots \widehat{f_{w_p}} \wedge f_n.
\end{aligned}$$

The two terms  $(-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}}$  (and with  $w_i < w_{p+2}$  respectively) appear since the original summand in  $\delta^{[2]}$  carries the sign  $(-1)^{i+j}$ , so we need to compute the number of the wedge slot correctly, respecting the omitted wedge factors; compare with the discussion in the first part of this proof. If the  $f_1, \dots, f_n$  pairwise commute, we see that the above expression is zero, so  $\delta^{[2]}$  makes no contribution and we are done. Hence, we may from now on assume our 2<sup>nd</sup> condition that the  $P_1^+, \dots, P_n^+$  commute pairwise. In view of the above, we next use the notation of eq. 4.4 and write the above in terms of

$$\begin{aligned}
(\delta^{[2]} \theta_{p+1,q})_{p+1,q-1|s_1 \dots s_n}^{w_1 \dots w_p \| w_{p+1}, w_{p+2}} &= (-1)^{w_{p+1} + w_{p+2}} (-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+1}\}} \\
&(-1)^{\#\{w_i | 1 \leq i \leq p \text{ s.t. } w_i < w_{p+2}\}} \theta_{p+1,q|s_1 \dots s_n}^{w_1 \dots w_p}.
\end{aligned}$$

Next, we apply  $H_{p+1}$  (see eq. 3.3 for the definition): Then for indices  $s_1 \dots s_n = \gamma_1^\dagger \dots \gamma_{n-p-1}^\dagger 0 \dots 0$  and  $\gamma_1^\dagger \dots \gamma_{n-p-1}^\dagger \in \{\pm\}$  (which is of degree  $p+2$ ) we obtain the expression

$$\begin{aligned}
&(H \delta^{[2]} \theta_{p+1,q})_{p+2,q-1|\gamma_1^\dagger \dots \gamma_{n-p-1}^\dagger \underbrace{0 \dots 0}_{p+1}}^{w_1 \dots w_p \| w_{p+1}, w_{p+2}} \\
&= P_1^{\gamma_1^\dagger} \dots P_{n-p-1}^{\gamma_{n-p-1}^\dagger} \sum_{\gamma_1, \dots, \gamma_{n-p} \in \{\pm\}} (-1)^{(\dots)} P_{n-p}^{-\gamma_{n-p}} \theta_{p+1,q|\gamma_1 \dots \gamma_{n-p}}^{w_1 \dots w_p \underbrace{0 \dots 0}_p}
\end{aligned}$$

where we have plugged in our previous computation and started to disregard the precise sign. We know the last term of this expression by our induction hypothesis and therefore obtain

$$\begin{aligned} &= P_1^{\gamma_1^\dagger} \dots P_{n-p-1}^{\gamma_{n-p-1}^\dagger} \\ &\sum_{\gamma_1, \dots, \gamma_{n-p} \in \{\pm\}} \sum_{\gamma_{n-p+1}^* \dots \gamma_n^* \in \{\pm\}} (-1)^{(\dots)} \underbrace{P_{n-p}^{-\gamma_{n-p}} P_1^{\gamma_1} \dots P_{n-p}^{\gamma_{n-p}}}_{\dots} \\ &\left( P_{n-p+1}^{(-\gamma_{n-p+1}^*)} \text{ad}(f_{w_p}) P_{n-p+1}^{\gamma_{n-p+1}^*} \right) \dots \left( P_n^{(-\gamma_n^*)} \text{ad}(f_{w_1}) P_n^{\gamma_n^*} \right) f_0. \end{aligned}$$

As the  $P_1^+, \dots, P_n^+$  commute pairwise, the same holds for all  $P_1^\pm, \dots, P_n^\pm$  (by Def. 3). Thus, the underlined expression can be rearranged to  $P_{n-p}^{-\gamma_{n-p}} P_{n-p}^{\gamma_{n-p}} \dots$ , but  $P_i^+ P_i^- = P_i^+ (\mathbf{1} - P_i^+) = 0$  as  $P_i^+$  is an idempotent. The same for  $P_i^- P_i^+$ . Hence, in all the indices  $s_1 \dots s_n$  relevant for our claim  $H\delta^{[2]} \theta_{p+1,q}$  is zero.  $\square$

This readily implies the following key computation:

**Theorem 6** (Main Theorem). *Let  $(A, (I_i^\pm), \tau)$  be an  $n$ -fold cubically decomposed algebra over a field  $k$ . Then*

$$\begin{aligned} &\text{res}_*(f_0 \wedge f_1 \wedge \dots \wedge f_n) \\ &= \tau \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} \text{ad } f_{\pi(1)} P_1^{\gamma_1}) \\ &\quad \dots (P_n^{-\gamma_n} \text{ad } f_{\pi(n)} P_n^{\gamma_n}) f_0, \end{aligned}$$

where  $P_1^+, \dots, P_n^+$  is any system of pairwise commuting good idempotents in the sense of eq. 3 (the value does not depend on the choice of the latter). Analogously,

$$(\text{res}^* \varphi)(f_0 \wedge f_1 \wedge \dots \wedge f_n) := \varphi \cdot \text{res}_*(f_0 \wedge \dots \wedge f_n)$$

for every  $\varphi \in k$ .

We remark that one can also write the above formula as

$$\begin{aligned} &\text{res}_*(f_0 \wedge f_1 \wedge \dots \wedge f_n) \\ &= \tau \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} f_{\pi(1)} P_1^{\gamma_1}) \dots (P_n^{-\gamma_n} f_{\pi(n)} P_n^{\gamma_n}) f_0 \end{aligned}$$

since for any expression  $g$  we have

$$\begin{aligned} (4.9) \quad &P_i^{-\gamma_i} \text{ad}(f_w) P_i^{\gamma_i} g = P_i^{-\gamma_i} [f_w, P_i^{\gamma_i} g] = P_i^{-\gamma_i} f_w P_i^{\gamma_i} g - P_i^{-\gamma_i} P_i^{\gamma_i} g f_w \\ &= P_i^{-\gamma_i} f_w P_i^{\gamma_i} g \end{aligned}$$

since  $P_i^{-\gamma_i} P_i^{\gamma_i} = (\mathbf{1} - P_i^{\gamma_i}) P_i^{\gamma_i} = 0$  and  $P_i^{\gamma_i}$  is an idempotent.

*Proof.* Use Prop. 1 with  $p = n$ . Plugging these components into the short-hand notation of eq. 4.3 we unwind for  $\text{res}_*(f_0 \wedge f_1 \wedge \dots \wedge f_n)$  the formula

$$\begin{aligned} &= \tau \sum_{\substack{w_1 \dots w_n \\ = \{1, \dots, n\}}} (-1)^n \rho(w_1, \dots, w_n) (-1)^{w_1 + \dots + w_n} \\ &\sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_{w_n}) P_1^{\gamma_1}) \dots (P_n^{-\gamma_n} \text{ad}(f_{w_1}) P_n^{\gamma_n}) f_0. \end{aligned}$$



We can clearly replace  $w_1, \dots, w_n$  by a sum over all permutations of  $\{1, \dots, n\}$ . In order to obtain a nice formula (in the above formula the  $P_i$  appear in ascending order, while the  $w_i$  appear in descending order), we prefer to compose each permutation with the order-reversing permutation  $w_i := \pi(n - i + 1)$ : Hence,

$$\begin{aligned} &= \tau \sum_{\pi \in \mathfrak{S}_n} (-1)^n \rho(\pi(n), \dots, \pi(1)) (-1)^{1+\dots+n} \\ &\quad \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} \text{ad}(f_{\pi(1)}) P_1^{\gamma_1}) \cdots (P_n^{-\gamma_n} \text{ad}(f_{\pi(n)}) P_n^{\gamma_n}) f_0. \end{aligned}$$

To conclude, use eq. 4.6 and the (easy) fact that the order-reversing permutation has signum  $(-1)^{\frac{(n-1)n}{2}}$  to see that

$$(-1)^n \rho(\pi(n), \dots, \pi(1)) (-1)^{1+\dots+n} = \text{sgn}(\pi).$$

□

*Proof of Thms. 1 & 2.* We define  $\mathfrak{G} := E^n(k)$ , where  $E$  is the functor defined in §1.1. As already discussed in §1.1 this contains  $k[t_1^\pm, \dots, t_n^\pm]$  as a Lie subalgebra, acting as multiplication operators  $x \mapsto f \cdot x$ . It is also easily checked that the differential operators  $t_1^{s_1} \cdots t_n^{s_n} \partial_{t_i}$  can be written as infinite matrices. If  $\mathfrak{g}$  is a *finite*-dimensional Lie algebra, observe that  $\mathfrak{G} = E^n(k)$  and  $E^n(\text{End}_k(\mathfrak{g}))$  are actually isomorphic. If  $\mathfrak{g}$  is simple, it is centreless, so the adjoint representation gives an embedding  $\mathfrak{g} \hookrightarrow \text{End}_k(\mathfrak{g})$ , and thus

$$\mathfrak{g}[t_1^\pm, \dots, t_n^\pm] \hookrightarrow E^n(\text{End}_k(\mathfrak{g}))_{Lie} \simeq E^n(k) = \mathfrak{G}.$$

This shows that all Lie algebras in the claim are subalgebras of  $\mathfrak{G}$ . As shown in §1.1,  $\mathfrak{G}$  is a cubically decomposed algebra, so we define  $\phi$  as in Def. 5. Using Thm. 6 we get an explicit formula for  $\phi$ , proving Thm. 2. Using the explicit formula, it is a direct computation to check that for  $n = 1$  the cocycle agrees with the ones mentioned in the claim of Thm. 1. □

## 5. APPLICATION TO THE MULTIDIMENSIONAL RESIDUE

In this section we will show that the Lie homology class of Def. 5 naturally gives the multidimensional (Parshin) residue.

We work in the framework of multivariate Laurent polynomial rings over a field  $k$ , see §1.1. In other words, as our cubically decomposed algebra we take an infinite matrix algebra  $A = E^n(k)$  and  $\mathfrak{g} = A_{Lie}$ . Via eq. 1.4 it acts on the  $k$ -vector space  $k[t_1^\pm, \dots, t_n^\pm]$ . The latter, now interpreted as a ring, also embeds as a *commutative* subalgebra into  $A$ . In order to distinguish very clearly between the subalgebra of  $A$  and the vector space it acts on, we shall from now on write  $k[\mathbf{t}_1^\pm, \dots, \mathbf{t}_n^\pm]$  for the  $k$ -vector space. Thus, when we write  $t_i$  we always refer to the associated multiplication operator  $x \mapsto t_i \cdot x$  in  $A$ , e.g.  $t_i^m \cdot \mathbf{t}_i^l = \mathbf{t}_i^{m+l}$ .

Following [2, Lemma 1(b)] we define a morphism of  $k$ -vector spaces

$$(5.1) \quad \varkappa : \Omega_{k[t_1^\pm, \dots, t_n^\pm]/k}^n \rightarrow H_{n+1}(\mathfrak{g}, k) \quad f_0 df_1 \wedge \dots \wedge df_n \mapsto f_0 \wedge f_1 \wedge \dots \wedge f_n.$$

As  $k[t_1^\pm, \dots, t_n^\pm]$  is commutative, the  $f_i$  commute pairwise and thus  $f_0 \wedge \dots \wedge f_n$  is indeed a Lie homology cycle.

**Theorem 7.** *The morphism*

$$\text{res}_* \circ \varkappa : \Omega_{k[t_1^\pm, \dots, t_n^\pm]/k}^n \longrightarrow k$$

(with  $\varkappa$  as in eq. 5.1 and  $\text{res}_*$  as in Def. 5) for  $c_{i,j} \in \mathbf{Z}$  is explicitly given by

$$t_1^{c_{0,1}} \dots t_n^{c_{0,n}} d(t_1^{c_{1,1}} \dots t_n^{c_{1,n}}) \wedge \dots \wedge d(t_1^{c_{n,1}} \dots t_n^{c_{n,n}}) \mapsto (-1)^n \det \begin{pmatrix} c_{1,1} & \dots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \dots & c_{n,n} \end{pmatrix}$$

whenever  $\sum_{p=0}^n c_{p,i} = 0$  and is zero otherwise. In particular  $(-1)^n (\text{res}_* \circ \varkappa)$  is the conventional multidimensional (Parshin) residue.

Letting  $c_{i,j} = \delta_{i=j}$  for  $i, j \in \{1, \dots, n\}$  gives the familiar

$$(-1)^n \text{res}_*(at_1^{c_{0,1}} \dots t_n^{c_{0,n}} \wedge t_1 \wedge \dots \wedge t_n) = \delta_{c_{0,1}=-1} \dots \delta_{c_{0,n}=-1} a$$

for  $a \in k$ . In particular this assures us that the map  $\text{res}_*$  gives the correct notion of residue: it is the  $(-1, \dots, -1)$ -coefficient of the Laurent expansion.

*Proof.* After unwinding  $\varkappa$  it remains to evaluate  $\text{res}_*(f_0 \wedge f_1 \wedge \dots \wedge f_n)$  for  $f_i := t_1^{c_{i,1}} \dots t_n^{c_{i,n}}$  ( $i = 0, \dots, n$ ) and by Thm. 6 this reduces to the matrix trace

$$(5.2) \quad \text{res}_*(f_0 \wedge f_1 \wedge \dots \wedge f_n) = \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) \tau M_\pi, \text{ where} \\ M_\pi := \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} (P_1^{-\gamma_1} f_{\pi(1)} P_1^{\gamma_1}) \dots (P_n^{-\gamma_n} f_{\pi(n)} P_n^{\gamma_n}) f_0.$$

For the evaluation of  $\tau M_\pi$  fix a permutation  $\pi$  and pick the (pairwise commuting) system of idempotents given by

$$(5.3) \quad P_j^+ \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} = \delta_{\lambda_j \geq 0} \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}. \quad (\text{with } \lambda_1, \dots, \lambda_n \in \mathbf{Z})$$

Next, observe that the Laurent polynomial ring  $W := k[\mathbf{t}_1^\pm, \dots, \mathbf{t}_n^\pm]$  is stable (i.e.  $\phi W \subseteq W$ ) under the endomorphisms  $f_0, \dots, f_n$  and the idempotents  $P_i^\pm$ , and therefore under  $M_\pi$ . Hence, it follows that it suffices to evaluate the trace of  $M_\pi$  on the  $k$ -vector subspace  $k[\mathbf{t}_1^\pm, \dots, \mathbf{t}_n^\pm]$ . We compute successively

$$\begin{aligned} f_k P_j^+ \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} &= \delta_{\lambda_j \geq 0} \mathbf{t}_1^{\lambda_1 + c_{k,1}} \dots \mathbf{t}_n^{\lambda_n + c_{k,n}} \\ P_j^- f_k P_j^+ \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} &= \delta_{0 \leq \lambda_j < -c_{k,j}} \mathbf{t}_1^{\lambda_1 + c_{k,1}} \dots \mathbf{t}_n^{\lambda_n + c_{k,n}} \end{aligned}$$

and analogously for  $P_j^+ f_k P_j^-$ . We find

$$(5.4) \quad \begin{aligned} \sum_{\gamma_j \in \{\pm\}} (-1)^{\gamma_j} \left( P_j^{-\gamma_j} f_k P_j^{\gamma_j} \right) \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} \\ = (\delta_{0 \leq \lambda_j < -c_{k,j}} - \delta_{-c_{k,j} \leq \lambda_j < 0}) \mathbf{t}_1^{\lambda_1 + c_{k,1}} \dots \mathbf{t}_n^{\lambda_n + c_{k,n}}. \end{aligned}$$

Now we claim:

- Subclaim: Writing  $w_i := \pi(i)$  we have

$$(5.5) \quad \begin{aligned} M_\pi \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} &= \prod_{i=1}^n (\delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < -c_{w_i,i}} \\ &\quad - \delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < 0}) \\ &\quad \mathbf{t}_1^{\lambda_1 + c_{0,1} + \sum_{p=1}^n c_{w_p,1}} \dots \mathbf{t}_n^{\lambda_n + c_{0,n} + \sum_{p=1}^n c_{w_p,n}}. \end{aligned}$$

(*Proof:* Define for  $i = 1, \dots, n+1$  the truncated sum

$$M_\pi^{(i)} := \left[ \sum_{\gamma_i \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_i + \dots + \gamma_n} (P_i^{-\gamma_i} f_{w_i} P_i^{\gamma_i}) \dots (P_n^{-\gamma_n} f_{w_n} P_n^{\gamma_n}) \right] f_0$$

so that  $M_\pi^{(1)} = M_\pi$  and  $M_\pi^{(n+1)} = f_0$ . We claim that

$$(5.6) \quad M_\pi^{(i)} \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} = \alpha \mathbf{t}_1^{\lambda_1 + c_{0,1} + \sum_{p=i}^n c_{w_p,1}} \dots \mathbf{t}_n^{\lambda_n + c_{0,n} + \sum_{p=i}^n c_{w_p,n}}$$

for some factor  $\alpha \in \{\pm 1, 0\}$ . For  $i = n+1$  this is clear since  $f_0 = t_1^{c_{0,1}} \dots t_n^{c_{0,n}}$ , in particular  $\alpha = 1$ . Assuming this holds for  $i+1$ , for  $i$  we get by using eq. 5.4 (with the appropriate values plugged in:  $j := i$  and  $k := w_i$ , and  $\lambda_i$  as in eq. 5.6)

$$(5.7) \quad \begin{aligned} M_\pi^{(i)} \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} &= \sum_{\gamma_i \in \{\pm\}} (-1)^{\gamma_i} (P_i^{-\gamma_i} f_{w_i} P_i^{\gamma_i}) M_\pi^{(i+1)} \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} \\ &= (\delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < -c_{w_i,i}} - \delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < 0}) \\ &\quad \alpha \mathbf{t}_1^{\lambda_1 + c_{0,1} + \sum_{p=i+1}^n c_{w_p,1} + c_{w_i,1}} \dots \mathbf{t}_n^{\lambda_n + c_{0,n} + \sum_{p=i+1}^n c_{w_p,n} + c_{w_i,n}}. \end{aligned}$$

This proves our claim for all  $i$  by induction. We observe that the pre-factor  $\alpha$  in each step just gets multiplied with the expression in eq. 5.7, giving the product in our claim.)

Next, we need to evaluate the trace of  $M_\pi$  as given in eq. 5.5. The endomorphism is nilpotent unless

$$(5.8) \quad \forall i : c_{0,1} + \sum_{p=1}^n c_{w_p,i} = 0.$$

We remark that  $w_1, \dots, w_n$  is just a permutation of  $\{1, \dots, n\}$ , so these conditions can be rewritten as  $\sum_{p=0}^n c_{p,i} = 0$ . In the nilpotent case the trace is clearly zero. Hence, we may assume we are in the case where eq. 5.8 holds. Using these equations and the useful convention  $w_{n+1} := 0$ , our expression for  $M_\pi$  simplifies to

$$(5.9) \quad \begin{aligned} M_\pi \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} &= \prod_{i=1}^n (\delta_{0 \leq \lambda_i + \sum_{p=i+1}^{n+1} c_{w_p,i} < -c_{w_i,i}} \\ &\quad - \delta_{0 \leq \lambda_i + c_{w_i,i} + \sum_{p=i+1}^{n+1} c_{w_p,i} < c_{w_i,i}}) \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}. \end{aligned}$$

The endomorphism  $M_\pi$  is visibly diagonal of finite rank and we may reduce the computation of the trace to a (finite-dimensional) stable vector subspace. A finite subset of the  $\mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}$  ( $\lambda_1, \dots, \lambda_n \in \mathbf{Z}$ ) provides a basis. We see in eq. 5.9 that  $M_\pi$  acts diagonally on these basis vectors with eigenvalues  $\pm 1$  or 0. Moreover, for each  $i$  we either have  $c_{w_i,i} \geq 0$  or  $c_{w_i,i} < 0$ , which shows that each bracket of the shape  $(\delta_{0 \leq \lambda < -c} - \delta_{-c \leq \lambda < 0})$  in eq. 5.9 either attains only values in  $\{+1, 0\}$  when we run through all  $\lambda_1, \dots, \lambda_n \in \mathbf{Z}$ , or only values in  $\{-1, 0\}$ . This shows that we only need to count (with appropriate sign) the non-zero eigenvalues of  $M_\pi$  in order to evaluate the trace. Note that our finite subset of  $\mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}$  ( $\lambda_1, \dots, \lambda_n \in \mathbf{Z}$ ) indexes a basis, so we need to count the number of such basis vectors with non-zero eigenvalue. We introduce the non-standard short-hand  $[x] := \min(0, x)$ . Inspecting eq. 5.9 shows that when running through  $\lambda_i$  we have

- $[-c_{w_i,i}]$  times the eigenvalue  $+1$ ,
- $[+c_{w_i,i}]$  times the eigenvalue  $-1$ .

The value of a fixed bracket  $(\delta_{0 \leq \lambda < -c} - \delta_{-c \leq \lambda < 0})$  - when non-zero - is always either  $+1$ , or always  $-1$ . Thus, the number of non-zero eigenvalues is simply the

number of elements within the hypercube such that each  $\lambda_i$  lies within the range of length  $\lfloor \pm c_{w_i, i} \rfloor$  counted above, and therefore

$$\tau M_\pi = \prod_{i=1}^n (\lfloor -c_{w_i, i} \rfloor - \lfloor +c_{w_i, i} \rfloor) = \prod_{i=1}^n (-c_{w_i, i}) = (-1)^n \prod_{i=1}^n c_{\pi(i), i}$$

(because  $\lfloor -a \rfloor - \lfloor a \rfloor = -a$  for all  $a \in \mathbf{Z}$ ). We plug this into eq. 5.2 and recognize the usual formula for the determinant. This finishes the proof.  $\square$

We are now ready to prove the theorem from the introduction:

*Proof of Thms. 4 & 5.* We use Thm. 7 to obtain **(2)**. Then **(3)** follows as a special case. For **(1)** use the short-hands  $\pi = P_1^+ = P^+$  (following both the notation of Arbarello, de Concini and Kac and ours). On the one hand we compute

$$\begin{aligned} [\pi, f_1]f_0 &= [P, f_1]f_0 = Pf_1f_0 - f_1Pf_0 = [Pf_0, f_1] \\ &= (P^+ + P^-)[P^+f_0, f_1] = P^-[P^+f_0, f_1] + P^+[P^+f_0, f_1] \end{aligned}$$

and we have  $[P^+f_0, f_1] + [P^-f_0, f_1] = [f_0, f_1] = 0$ , so this equals

$$= P^-[P^+f_0, f_1] - P^+[P^-f_0, f_1].$$

On the other hand, we unwind

$$\begin{aligned} \text{res } f_0 df_1 &= (-1)^1 \text{tr} \sum_{\gamma_1 \in \{\pm\}} (-1)^{\gamma_1} (P_1^{-\gamma_1} \text{ad}(f_{\pi(1)}) P_1^{\gamma_1}) f_0 \\ &= -P^-[f_1, P_1^+ f_0] + P^+[f_1, P_1^- f_0] \end{aligned}$$

and these expressions clearly coincide. Finally Thm. 5 is true since we use the cocycle defined in Def. 5, i.e. it is constructed exactly as stated in Thm. 5.  $\square$

## 6. APPLICATION TO MULTILOOP LIE ALGEBRAS

Suppose  $k$  is a field and  $\mathfrak{g}/k$  is a finite-dimensional centreless Lie algebra (e.g.  $\mathfrak{g}$  finite-dimensional, semisimple). Then the adjoint representation  $\text{ad} : \mathfrak{g} \hookrightarrow \text{End}_k(\mathfrak{g})$  is injective. Thus, we obtain a Lie algebra inclusion

$$i : \mathfrak{g}[\mathfrak{t}_1^\pm, \dots, \mathfrak{t}_n^\pm] \hookrightarrow E^n(\text{End}_k(\mathfrak{g}))_{\text{Lie}},$$

where  $E$  is the functor described in §1.1 (the right-hand side is equipped with the Lie bracket  $[a, b] = ab - ba$  based on the associative algebra structure). Thus, we have the pullback

$$i^* : H^{n+1}(E^n(\text{End}_R(\mathfrak{g}))_{\text{Lie}}, k) \rightarrow H^{n+1}(\mathfrak{g}[\mathfrak{t}_1^\pm, \dots, \mathfrak{t}_n^\pm], k),$$

which we may apply to the class  $\text{res}^*(1)$ , see Def. 5.

**Theorem 8.** *Suppose  $k$  is a field and  $\mathfrak{g}/k$  is a finite-dimensional centreless Lie algebra. For  $Y_0, \dots, Y_n \in \mathfrak{g}$  we call*

$$(6.1) \quad B(Y_0, \dots, Y_n) := \text{tr}_{\text{End}_k(\mathfrak{g})}(\text{ad}(Y_0) \text{ad}(Y_1) \cdots \text{ad}(Y_n))$$

*the ‘generalized Killing form’. For  $n = 1$  and if  $\mathfrak{g}$  is semisimple, this is the classical Killing form of  $\mathfrak{g}$ .*

- (1) *For all  $n \geq 1$  the pullback  $i^* \text{res}^*(1) \in H^{n+1}(\mathfrak{g}[\mathfrak{t}_1^\pm, \dots, \mathfrak{t}_n^\pm], k)$  is explicitly given by*

$$\begin{aligned} &(i^* \phi)(Y_0 \mathfrak{t}_1^{c_{0,1}} \cdots \mathfrak{t}_n^{c_{0,n}} \wedge \cdots \wedge Y_n \mathfrak{t}_1^{c_{n,1}} \cdots \mathfrak{t}_n^{c_{n,n}}) \\ &= (-1)^n \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) B(Y_{\pi(1)}, \dots, Y_{\pi(n)}, Y_0) \prod_{i=1}^n c_{\pi(i), i}. \end{aligned}$$

whenever  $\forall i \in \{1, \dots, n\} : \sum_{p=0}^n c_{p,i} = 0$  and zero otherwise.

- (2) If  $\mathfrak{g}$  is finite-dimensional and semisimple and  $n = 1$ , then  $i^* \text{res}^*(1) \in H^2(\mathfrak{g}[\mathbf{t}_1^\pm], k)$  is the universal central extension of the loop Lie algebra  $\mathfrak{g}[\mathbf{t}_1, \mathbf{t}_1^{-1}]$  giving the associated affine Lie algebra  $\widehat{\mathfrak{g}}$  (without extending by a derivation),

$$0 \longrightarrow k \langle c \rangle \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g}[\mathbf{t}_1, \mathbf{t}_1^{-1}] \longrightarrow 0.$$

*Proof.* (1) According to Thm. 6 and eq. 3.8 the cocycle is explicitly given by

$$\begin{aligned} \text{res}^*(1)(f_0 \wedge \dots \wedge f_n) &= \tau \sum_{\pi \in \mathfrak{S}_n} \text{sgn}(\pi) M_\pi, \text{ where} \\ M_\pi &= \sum_{\gamma_1 \dots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \dots + \gamma_n} \\ &\quad (P_1^{-\gamma_1} f_{\pi(1)} P_1^{\gamma_1}) \dots (P_n^{-\gamma_n} f_{\pi(n)} P_n^{\gamma_n}) f_0. \end{aligned}$$

Note that  $M_\pi \in E^n(\text{End}_k(\mathfrak{g}))$ . As we consider the pullback of the cohomology class along  $i : \mathfrak{g}[t_1^\pm, \dots, t_n^\pm] \hookrightarrow E^n(\text{End}_k(\mathfrak{g}))_{Lie}$ , it suffices to treat elements  $f_i := Y_i t_1^{c_{i,1}} \dots t_n^{c_{i,n}}$  with  $c_{i,1}, \dots, c_{i,n} \in \mathbf{Z}$  (for  $i = 0, \dots, n$ ) and  $Y_i \in \mathfrak{g}$ . Note that by our embedding  $i$  an element  $f_i$  is mapped to the endomorphism  $\text{ad}(Y_i) t_1^{c_{i,1}} \dots t_n^{c_{i,n}}$  in  $E^n(\text{End}_k(\mathfrak{g}))$ . Let  $\pi \in \mathfrak{S}_n$  be a fixed permutation. In order to compute the trace, it suffices to study the action of  $M_\pi$  on the basis elements  $X \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}$  of  $\mathfrak{g}[t_1^\pm, \dots, t_n^\pm]$ , where  $\lambda_1, \dots, \lambda_n \in \mathbf{Z}$  and  $X \in \mathfrak{g}$  runs through a basis of  $\mathfrak{g}$ . We denote them with bold letters  $\mathbf{t}_i$  instead of  $t_i$  to distinguish clearly between a basis element and  $t_i$  as an endomorphism  $t_i : x \mapsto t_i \cdot x$  in  $E^n(\text{End}_k(\mathfrak{g}))$ . As in the proof of Thm. 7 we compute

$$P_j^- f_k P_j^+ X \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} = \delta_{0 \leq \lambda_j < -c_{k,j}} \text{ad}(Y_k) X \mathbf{t}_1^{\lambda_1 + c_{k,1}} \dots \mathbf{t}_n^{\lambda_n + c_{k,n}}.$$

and as a consequence we find

$$\begin{aligned} \sum_{\gamma_j \in \{\pm\}} (-1)^{\gamma_j} (P_j^{-\gamma_j} x_k P_j^{\gamma_j}) X \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} \\ = (\delta_{0 \leq \lambda_j < -c_{k,j}} - \delta_{-c_{k,j} \leq \lambda_j < 0}) \text{ad}(Y_k) X \mathbf{t}_1^{\lambda_1 + c_{k,1}} \dots \mathbf{t}_n^{\lambda_n + c_{k,n}}. \end{aligned}$$

With an inductive computation entirely analogous to eq. 5.5 we find

$$\begin{aligned} M_\pi X \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n} &= \prod_{i=1}^n (\delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < -c_{w_i,i}} \\ &\quad - \delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < 0}) \\ &\quad \text{ad}(Y_{w_1}) \dots \text{ad}(Y_{w_n}) \text{ad}(Y_0) X \\ &\quad \mathbf{t}_1^{\lambda_1 + \sum_{p=0}^n c_{p,1}} \dots \mathbf{t}_n^{\lambda_n + \sum_{p=0}^n c_{p,n}}, \end{aligned}$$

where  $w_i := \pi(i)$ . Unless  $\forall i : \sum_{p=0}^n c_{p,i} = 0$  holds,  $M_\pi$  is clearly nilpotent and thus has trace  $\tau M_\pi = 0$ . This condition is clearly independent of  $\pi$ , showing that  $(i^* \text{res}^*(1))(f_0 \wedge \dots \wedge f_n) = 0$  in this case. From now on assume  $\forall i : \sum_{p=0}^n c_{p,i} = 0$ . Then  $M_\pi$  respects the decomposition

$$\mathfrak{g}[\mathbf{t}_1^\pm, \dots, \mathbf{t}_n^\pm] = \coprod_{\lambda_1, \dots, \lambda_n \in \mathbf{Z}^n} \mathfrak{g} \mathbf{t}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}$$

and therefore (as  $\tau$  is essentially a trace)  $\tau M_\pi = \sum_{\lambda_1, \dots, \lambda_n} \tau M_\pi|_{\mathbf{gt}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}}$ . For each summand of the latter we obtain

$$\begin{aligned} \tau M_\pi|_{\mathbf{gt}_1^{\lambda_1} \dots \mathbf{t}_n^{\lambda_n}} &= \prod_{i=1}^n (\delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < -c_{w_i,i}} \\ &\quad - \delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < 0}) \\ &\quad \text{tr}(\text{ad}(Y_{w_1}) \cdots \text{ad}(Y_{w_n}) \text{ad}(Y_0)). \end{aligned}$$

The trace term is independent of  $\lambda_1, \dots, \lambda_n$  (and in the shape of eq. 6.1), so we may rewrite  $\tau M_\pi$  as

$$\begin{aligned} \tau M_\pi &= B(Y_{w_1}, \dots, Y_{w_n}, Y_0) \sum_{\lambda_1, \dots, \lambda_n} \prod_{i=1}^n (\delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < -c_{w_i,i}} \\ &\quad - \delta_{-c_{w_i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{w_p,i} < 0}). \end{aligned}$$

For the evaluation of the sum  $\sum_{\lambda_1, \dots, \lambda_n}$  we can apply the same eigenvalue count as in the proof of Thm. 7. This time instead of counting eigenvalues, we count non-zero summands. This yields

$$\tau M_\pi = (-1)^n B(Y_{w_1}, \dots, Y_{w_n}, Y_0) \prod_{i=1}^n c_{w_i,i}$$

and thus our claim. **(2)** For  $n = 1$  we obtain

$$(i^* \text{res}^*(1))(Y_0 \mathbf{t}_1^{c_{0,1}} \wedge Y_1 \mathbf{t}_1^{c_{1,1}}) = -c_{1,1} \delta_{c_{0,1} + c_{1,1} = 0} B(Y_1, Y_0).$$

This is well-known to be the defining cocycle of the affine Lie algebra  $\widehat{\mathfrak{g}}$  (usually with a positive sign, but the class is only well-defined up to non-zero scalar multiple anyway).  $\square$

The natural further cases of the Virasoro algebra as well as affine Kac-Moody algebras (i.e.  $\widehat{\mathfrak{g}}$  extended by derivations) will be discussed elsewhere. The computations become more involved, but no further ideas are needed.

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